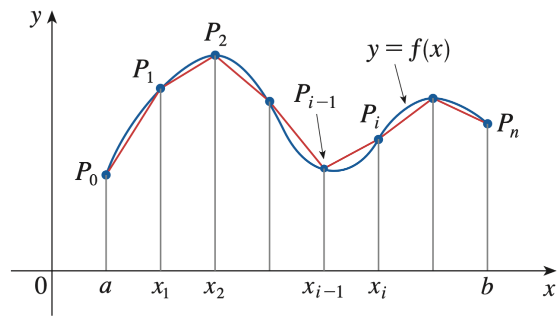
8.1 / 10.2 Arc Length

Theorem (Arc Length Formula)**.** Let $y = f(x)$ be a curve defined on the interval $[a, b]$, and suppose that $f'(x)$ is continuous on $[a, b]$. The arc length L of the curve is given by:

1. **Polygonal Approximation:**

   - Divide the interval $[a, b]$ into n subintervals of equal width Δx.
   - For each i, let $y_i = f(x_i)$ and consider the points $P_i = (x_i, y_i)$ on the curve.
   - Approximate the curve by a polygonal path connecting these points.



2. **Length of a Single Segment:**

   - The length of a segment connecting two consecutive points P_{i-1} and P_i is:

   - By the Mean Value Theorem, $\Delta y_i = f'(x_i^*)\Delta x$ for some $x_i^* \in [x_{i-1}, x_i]$.
   - Substitute this into the segment length:

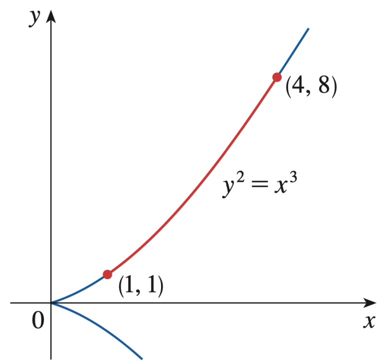
3. **Total Length of the Polygonal Path:**

   - Sum the lengths of all segments:

4. **Take the Limit as $n \to \infty$:**

   - As $n \to \infty$, $\Delta x \to 0$, and the sum becomes a definite integral:

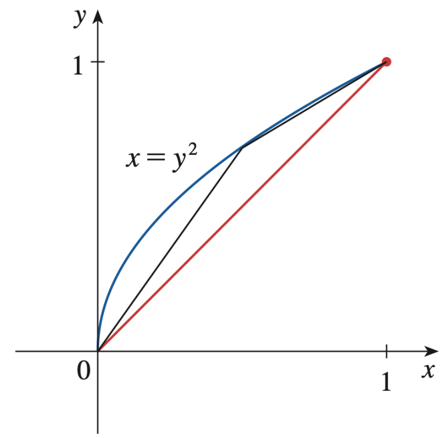
Example. Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$.



Theorem. If a curve has the equation $x = g(y)$, $c \leq y \leq d$, and $g'(y)$ is continuous, then by interchanging the roles of x and y, we obtain the following formula for its length:

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

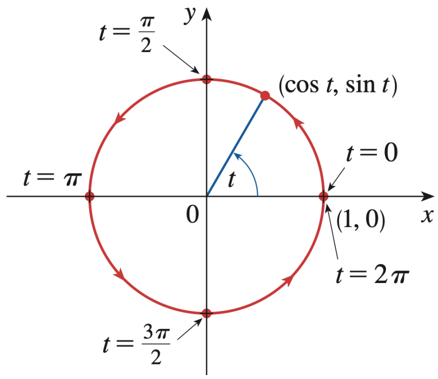
Example. Find the length of the arc of the parabola $x = y^2$ from $(0, 0)$ to $(1, 1)$.



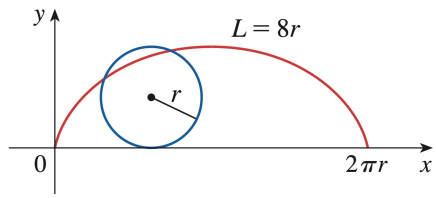
Example. Set up an integral for the length of the arc of the hyperbola $xy = 1$ from $(1, 1)$ to $(2, \frac{1}{2})$.

Theorem. If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where $f'(t)$ and $g'(t)$ are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is:

Example. Find the length of the unit circle described by $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$.



Example. Find the length of one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.



- One arch of the cycloid corresponds to the parameter interval: ________________ .

- Compute the derivatives:

- Substitute into the arc length formula:

- Use the trigonometric identity $1 - \cos\theta = 2\sin^2(\theta/2)$:

- Since $0 \le \theta \le 2\pi$, we have $0 \le \theta/2 \le \pi$, and so $\sin(\theta/2) \ge 0$.

Definition. Suppose a curve is described by $x = f(u)$, $y = g(u)$, where $f'(u)$ and $g'(u)$ are continuous. The arc length function $s(t)$ gives the length of a curve from an initial point $(f(a), g(a))$ to the point $(f(t), g(t))$ corresponding to the parameter t. In particular,

Remark. If parametric equations describe the position of a moving particle, then the speed $v(t)$ is the derivative of the arc length function $s(t)$. Indeed, the speed $v(t)$ is the rate of change of the total distance traveled along a curve with respect to time. By the Fundamental Theorem of Calculus,

Example. A particle's position is given by $x = 2t + 3$, $y = 4t^2$, $t \geq 0$. Find the speed of the particle when it is at the point $(5, 4)$.